

# COHOMOLOGY JUMP LOCI OF QUASI-PROJECTIVE VARIETIES

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ABSTRACT. We prove that the cohomology jump loci in the space of rank one local systems over a smooth quasi-projective variety are finite unions of torsion translates of subtori. The main ingredients are a recent result of Dimca-Papadima, some techniques introduced by Simpson, together with properties of the moduli space of logarithmic connections constructed by Nitsure and Simpson.

## 1. INTRODUCTION

Let  $X$  be a connected, finite-type CW-complex. Define

$$\mathbf{M}_B(X) = \text{Hom}(\pi_1(X), \mathbb{C}^*)$$

to be the variety of  $\mathbb{C}^*$  representations of  $\pi_1(X)$ . Then  $\mathbf{M}_B(X)$  is a direct product of  $(\mathbb{C}^*)^{b_1(X)}$  and a finite abelian group. For each point  $\rho \in \mathbf{M}_B(X)$ , there exists a unique rank one local system  $L_\rho$ , whose monodromy representation is  $\rho$ . The *cohomology jump loci* of  $X$  are the natural strata

$$\Sigma_k^i(X) = \{\rho \in \mathbf{M}_B(X) \mid \dim_{\mathbb{C}} H^i(X, L_\rho) \geq k\}.$$

$\Sigma_k^i(X)$  is a Zariski closed subset of  $\mathbf{M}_B(X)$ . A celebrated result of Simpson says that if  $X$  is a smooth projective variety defined over  $\mathbb{C}$ , then  $\Sigma_k^i(X)$  is a union of torsion translates of subtori of  $\mathbf{M}_B(X)$ .

In this paper, we generalize Simpson's result to quasi-projective varieties.

**Theorem 1.1.** *Suppose  $U$  is a smooth quasi-projective variety defined over  $\mathbb{C}$ . Then  $\Sigma_k^i(U)$  is a finite union of torsion translates of subtori of  $\mathbf{M}_B(U)$ .*

When  $U$  is compact, the theorem is proved in [GL1, GL2], [A1], [S2], with the strongest form appearing in the latter. When  $b_1(\bar{U}) = 0$ , Arapura [A2] showed that  $\Sigma_k^i(U)$  are union of translates of subtori. The case of unitary rank one local systems on  $U$  has been considered in [B] and [L]. Libgober [L] also proved the same theorem for  $U = \mathcal{X} - \mathcal{D}$  where  $\mathcal{X}$  is a germ of a smooth analytic space, and  $\mathcal{D}$  is a divisor of  $\mathcal{X}$ . By studying Morgan's Gysin model, Dimca and Papadima were able to prove the following:

**Theorem 1.2.** [DP, Theorem C] *Under the same assumption as Theorem 1.1, every irreducible component of  $\Sigma_k^i(U)$  containing  $1 \in \mathbf{M}_B(U)$  is a subtorus.*

In fact, this result of Dimca and Papadima serves as a key ingredient of our theorem. In Section 2, we will show that each irreducible component of  $\Sigma_k^i(U)$  contains a torsion point. Then, in Section 3, we will see that, thanks to Theorem 1.2, having a torsion point on an irreducible component of  $\Sigma_k^i(U)$  forces this component to be a translate of subtorus.

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## 2. TORSION POINTS ON THE COHOMOLOGY JUMP LOCI

Let  $X$  be a smooth complex projective variety, and let  $D = \sum_{\lambda=1}^n D_\lambda$  be a simple normal crossing divisor on  $X$  with irreducible components  $D_\lambda$ . Let  $U = X - D$ . Thanks to Hironaka's theorem on resolution of singularities, every smooth quasi-projective variety  $U$  can be realized in this way. The goal of this section is to prove the following:

**Theorem 2.1.** *Each irreducible component of  $\Sigma_k^i(U)$  contains a torsion point.*

First, we want to reduce to the case when  $X$  and each  $D_\lambda$  are defined over  $\bar{\mathbb{Q}}$ . This can be done using a technique which we have learnt from the proof of [S2, Theorem 4.1]. We reproduce it here.

We can assume  $X$  and each  $D_\lambda$  to be defined over a subring  $O$  of  $\mathbb{C}$ , which is finitely generated over  $\mathbb{Q}$ . Denote the embedding of  $O$  to  $\mathbb{C}$  by  $\sigma : O \rightarrow \mathbb{C}$ . Each ring homomorphism  $O \rightarrow \mathbb{C}$  corresponds to a point in  $\text{Spec}(O)(\mathbb{C})$ . Denote by  $X^0$  and  $D_\lambda^0$  the schemes over  $\text{Spec}(O)$  which give rise to  $X$  and  $D_\lambda$  respectively after tensoring with  $\mathbb{C}$ , that is  $X = X^0 \times_{\text{Spec}(O)} \text{Spec}(\mathbb{C})$  and  $D_\lambda = D_\lambda^0 \times_{\text{Spec}(O)} \text{Spec}(\mathbb{C})$ . By possibly replacing  $O$  by  $O[\frac{1}{h}]$  for some  $h \in O$ , we can assume  $X^0$  and every  $D_\lambda^0$  are smooth over  $\text{Spec}(O)$ , and all the intersections of  $D_\lambda^0$ 's are transversal. Since each connected component of  $\text{Spec}(O)(\mathbb{C})$  contains a  $\bar{\mathbb{Q}}$  point, there exists a point  $P \in \text{Spec}(O)(\bar{\mathbb{Q}})$ , and a continuous path from  $\sigma \in \text{Spec}(O)(\mathbb{C})$  to  $P$  in  $\text{Spec}(O)(\mathbb{C})^{\text{top}}$ . Then, according to Thom's First Isotopy Lemma [Di, Ch. 1, Theorem 3.5],  $X^0(\mathbb{C})$  together with its strata given by the  $D_\lambda^0(\mathbb{C})$ , is a topologically locally trivial fibration in the stratified sense over  $\text{Spec}(O)(\mathbb{C})^{\text{top}}$ . In particular, letting  $X'$  and  $D'_\lambda$  be the corresponding fibers over  $P$ , transporting along the path gives an isomorphism  $(X - D)^{\text{top}} \cong (X' - D')^{\text{top}}$ . Recall that  $\mathbf{M}_B(U)$  and  $\Sigma_k^i(U)$  depend only on the topology of  $U$ . Hence replacing  $U = X - D$  by  $U' = X' - D'$ , we may assume that  $X$  and each  $D_\lambda$  are defined over  $\mathbb{Q}$ .

Next, we introduce the other side of the story, namely the logarithmic flat bundles on  $(X, D)$ . A logarithmic flat bundle on  $(X, D)$  consists of a vector bundle  $E$  on  $X$ , and a logarithmic connection  $\nabla : E \rightarrow E \otimes \Omega_X^1(\log D)$ , satisfying the integrability condition  $\nabla^2 = 0$ . Given a logarithmic flat bundle  $(E, \nabla)$ , the flat sections of  $E$  on  $U$  (by which we will always mean on  $U^{\text{top}}$ ) form a local system. And conversely, given any local system  $L$  on  $U$  (by which, as in the introduction, we will always mean a local system on  $U^{\text{top}}$ ), it is always obtained from some logarithmic flat bundle  $(E, \nabla)$ . However, different logarithmic flat bundles may give the same local system. This correspondence between local systems on  $U$  and logarithmic flat bundles on  $(X, D)$  is very well understood (e.g. [D], [S1], [M]).

For a vector bundle  $E$  on  $X$ , the structure of a logarithmic flat bundle  $(E, \nabla)$  on  $(X, D)$  is the same as a  $\mathcal{D}_X(\log D)$ -module structure on  $E$ , where  $\mathcal{D}_X(\log D)$  is the sheaf of logarithmic differentials.

Nitsure [N] and Simpson [S3] constructed coarse moduli spaces, which are separated quasi-projective schemes, for Jordan-equivalence classes of semistable  $\Lambda$ -modules which are  $\mathcal{O}_X$ -coherent and torsion free, where  $\Lambda$  is a sheaf of rings of differential operators. The two examples of  $\Lambda$  which we are concerned with are  $\mathcal{D}_X$ , the usual sheaf of differential

operators on  $X$ , and  $\mathcal{D}_X(\log D)$ , the sheaf of logarithmic differentials. We denote by  $\mathbf{M}_{\text{DR}}(X)$  and  $\mathbf{M}_{\text{DR}}(X/D)$  the moduli space of rank one  $\mathcal{D}_X$ -modules and the moduli space of rank one  $\mathcal{D}_X(\log D)$ -modules, respectively. In the rank one case, semistable is the same as stable and this condition is automatic as is the locally free condition, and Jordan-equivalence is the same as isomorphic. Thus, the points of  $\mathbf{M}_{\text{DR}}(X)$  and  $\mathbf{M}_{\text{DR}}(X/D)$  correspond to isomorphism classes of flat, respectively, logarithmic flat line bundles. Since we did not put any condition on the Chern class of the underlying line bundles, in general  $\mathbf{M}_{\text{DR}}(X/D)$  has infinitely many connected components.  $\mathbf{M}_{\text{DR}}(X)$ ,  $\mathbf{M}_{\text{DR}}(X/D)$ ,  $\mathbf{M}_B(X)$  and  $\mathbf{M}_B(U)$  are all algebraic groups, except  $\mathbf{M}_{\text{DR}}(X/D)$  is not of finite type.

The following diagram plays an essential role in our proof.

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \mathbb{Z}^n & \xlongequal{\quad} & \mathbb{Z}^n & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{M}_{\text{DR}}(X) & \longrightarrow & \mathbf{M}_{\text{DR}}(X/D) & \xrightarrow{\text{res}} & \mathbb{C}^n \\
& & \downarrow \text{RH} & & \downarrow \text{RH} & & \downarrow \text{exp} \\
0 & \longrightarrow & \mathbf{M}_B(X) & \longrightarrow & \mathbf{M}_B(U) & \xrightarrow{\text{ev}} & (\mathbb{C}^*)^n \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 
\end{array}$$

Let us first explain how the arrows are defined. Since every  $\mathcal{D}_X$ -module is naturally a  $\mathcal{D}_X(\log D)$ -module, there is a natural embedding  $\mathbf{M}_{\text{DR}}(X) \hookrightarrow \mathbf{M}_{\text{DR}}(X/D)$ . On the other hand, the embedding  $U \hookrightarrow X$  induces a surjective map on the fundamental group  $\pi_1(U) \rightarrow \pi_1(X)$ . Composing this map with the representations, we have  $\mathbf{M}_B(X) \hookrightarrow \mathbf{M}_B(U)$ . For every rank one logarithmic flat bundle  $(E, \nabla)$ , taking the residue along each  $D_\lambda$  is the map  $\text{res}$ . In other words,  $\text{res}((E, \nabla)) = \{\text{res}_{D_\lambda}(\nabla)\}_{1 \leq \lambda \leq n}$ . Around each  $D_\lambda$ , we can take a small loop  $\gamma_\lambda$ . The map  $\text{ev}$  is the evaluation at the loops  $\gamma_\lambda$ . More precisely  $\text{ev}(\rho) = \{\rho(\gamma_\lambda)\}_{1 \leq \lambda \leq n}$ .

For the horizontal arrows,  $\text{RH} : \mathbf{M}_{\text{DR}}(X) \rightarrow \mathbf{M}_B(X)$  is taking the monodromy representations for flat bundles. Since every logarithmic flat bundle on  $(X, D)$  restricts to a flat bundle on  $U$ , taking the monodromy representation on  $U$  is  $\text{RH} : \mathbf{M}_{\text{DR}}(X/D) \rightarrow \mathbf{M}_B(U)$ . The map  $\text{exp} : \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$  is component-wise defined to be multiplying by  $2\pi\sqrt{-1}$ , then taking exponential. On  $\mathbf{M}_{\text{DR}}(X/D)$ , there are some special elements. Let  $(\mathcal{O}_X, d)$  be the trivial rank one logarithmic flat bundle on  $(X, D)$ . Notice that  $\mathcal{O}_X(-D_\lambda)$  is preserved under  $d$ , that is, there is an induced map  $d : \mathcal{O}_X(-D_\lambda) \rightarrow \mathcal{O}_X(-D_\lambda) \otimes \Omega_X^i(\log D)$ . Therefore,  $(\mathcal{O}_X(-D_\lambda), d)$  is also a logarithmic flat bundle on  $(X, D)$ . The map  $\mathbb{Z}^n \rightarrow \mathbf{M}_{\text{DR}}(X/D)$  is defined by  $\{m_\lambda\}_{1 \leq \lambda \leq n} \mapsto \bigotimes_{1 \leq \lambda \leq n} (\mathcal{O}_X(-D_\lambda), d)^{\otimes m_\lambda}$ . The map  $\mathbb{Z}^n \rightarrow \mathbb{C}^n$  is the natural inclusion map.

Notice that all the maps are group homomorphisms, all the rows and columns are exact. It is proved in [S4] that the first map  $\text{RH}$  is an analytic isomorphism. By a

straightforward modification, one can also prove that the second map  $RH$  is an analytic covering map. The map  $\exp$  is obviously an analytic covering map.

According to the discussion following Theorem 2.1, we can assume  $X$  and each  $D_\lambda$  to be defined over  $\bar{\mathbb{Q}}$  without loss of generality. Then  $\mathbf{M}_{\text{DR}}(X/D)$  and  $\mathbf{M}_{\text{DR}}(X)$  are also defined over  $\bar{\mathbb{Q}}$ . The representation varieties  $\mathbf{M}_B(U)$  and  $\mathbf{M}_B(X)$  are always defined over  $\mathbb{Q}$ . Therefore, all the horizontal arrows in the above diagram are maps defined over  $\bar{\mathbb{Q}}$ . From now on, we should think of  $\mathbb{C}^n$  and  $(\mathbb{C}^*)^n$  as varieties defined over  $\bar{\mathbb{Q}}$ , or in other words, as  $A_C^n = A_{\bar{\mathbb{Q}}}^n \times_{\bar{\mathbb{Q}}} \mathbb{C}$  and  $(G_{m,C})^n = (G_{m,\bar{\mathbb{Q}}})^n \times_{\bar{\mathbb{Q}}} \mathbb{C}$ , respectively.

**Lemma 2.2.** *Suppose  $Z \subset \mathbb{C}^n$  is a closed algebraic variety defined over  $\bar{\mathbb{Q}}$ . And suppose  $\exp(Z) \subset (\mathbb{C}^*)^n$  is also a closed algebraic variety defined over  $\bar{\mathbb{Q}}$ . Then  $\exp(Z)$  contains a torsion point.*

*Proof.* When  $n = 1$ , this is the Gelfond-Schneider theorem, which says if  $\alpha$  and  $e^{2\pi\sqrt{-1}\alpha}$  are both algebraic numbers, then  $\alpha \in \mathbb{Q}$ .

We use induction on  $n$ . Suppose the lemma is true for  $\mathbb{C}^{n-1}$ . Let  $p_1 : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$  and  $p_2 : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^{n-1}$  be the projections to the first  $n - 1$  factors. Then  $p_1(Z) \subset \mathbb{C}^{n-1}$  and  $p_2(\exp(Z)) \subset (\mathbb{C}^*)^{n-1}$  are both defined over  $\bar{\mathbb{Q}}$ . Since  $\exp(p_1(Z)) = p_2(\exp(Z))$ , by induction hypothesis,  $\exp(p_1(Z))$  contains a torsion point  $\tau$  in  $(\mathbb{C}^*)^{n-1}$ .

Let  $M = p_2^{-1}(\tau)$ . Then  $\exp^{-1}(M)$  is a disjoint union of infinitely many copies of  $\mathbb{C}$ . Choose one copy of those, which intersects with  $Z$ . Denote this copy by  $N$ . Since  $\tau$  is a torsion point in  $(\mathbb{C}^*)^{n-1}$ ,  $N$  is defined by equations with  $\mathbb{Q}$  coefficients. Consider the following diagram,

$$\begin{array}{ccc} N & \xrightarrow{q_1} & \mathbb{C} \\ \exp \downarrow & & \downarrow \exp \\ M & \xrightarrow{q_2} & \mathbb{C}^* \end{array}$$

where  $q_1$  and  $q_2$  are projections to the last coordinates respectively. Then  $q_1$  and  $q_2$  are isomorphisms defined over  $\bar{\mathbb{Q}}$ . If  $M \subset \exp(Z)$ , then every torsion point in  $\mathbb{C}^*$  via  $q_2^{-1}$  gives a torsion point in  $\exp(Z)$ . If  $M \not\subset \exp(Z)$ , then  $M \cap \exp(Z)$  contains finitely many points. Hence,  $N \cap Z$  also contains finitely many points. In this case, let  $\sigma$  be any point in  $N \cap Z$ ,  $q_1(\sigma) \in \mathbb{C}$  is defined over  $\bar{\mathbb{Q}}$ . On the other hand,  $\exp(\sigma)$  is a point in  $M \cap \exp(Z)$ , and hence defined over  $\bar{\mathbb{Q}}$ . Thus,  $q_2(\exp(\sigma)) = \exp(q_1(\sigma)) \in \mathbb{C}^*$  is defined over  $\bar{\mathbb{Q}}$ . Now, the Gelfond-Schneider theorem implies that  $p_2(\exp(\sigma))$  is torsion in  $\mathbb{C}^*$ . Since  $q_2(\exp(\sigma))$  is torsion in  $\mathbb{C}^*$  and  $p_2(\exp(\sigma)) = \tau$  is torsion in  $(\mathbb{C}^*)^{n-1}$ ,  $\exp(\sigma) \in \exp(Z)$  is a torsion point in  $(\mathbb{C}^*)^n$ .  $\square$

**Remark 2.3.** In fact, using Hilbert's irreducibility theorem, one can prove that  $\exp(Z)$  is a finite union of torsion translates of subtori. This was pointed out to us by Jiu-Kang Yu.

**Lemma 2.4.** *Let  $T$  be an irreducible component of  $\Sigma_k^i(U)$ . Then there exists an irreducible subvariety  $S$  of  $\mathbf{M}_{\text{DR}}(X/D)$  defined over  $\bar{\mathbb{Q}}$  such that  $RH(S) = T$ .*

*Proof.* For any  $\rho \in \mathbf{M}_B(U)$ ,  $RH^{-1}(\rho)$  contains all the possible extensions of  $L_\rho$  to a logarithmic flat bundle over  $(X, D)$ . Suppose  $(E, \nabla) \in RH^{-1}(L)$ , and suppose  $\nabla$  does not have any residue being equal to a positive integer, that is,  $\text{res}((E, \nabla))$  does not have

any positive integer in its coordinates. Then by a theorem of Deligne [D, II, 6.10], the hypercohomology of the algebraic de Rham complex

$$E \otimes \Omega_X^\bullet(\log D) = [E \xrightarrow{\nabla} E \otimes \Omega_X^1(\log D) \xrightarrow{\nabla} E \otimes \Omega_X^2(\log D) \xrightarrow{\nabla} \cdots]$$

computes the cohomology of the local system  $L$ , i.e.,  $H^i(X, E \otimes \Omega_X^\bullet(\log D)) \cong H^i(U, L_\rho)$ .

Define the bad locus  $\mathbf{BL} \subset \mathbf{M}_{\text{DR}}(X/D)$  to be the locus where one of the residues of  $\nabla$  is a positive integer. Then  $\mathbf{BL}$  is the preimage of infinitely many hyperplanes in  $\mathbb{C}^n$  via  $\text{res}$ . Define

$$\Sigma_k^i(X/D) = \{(E, \nabla) \in \mathbf{M}_{\text{DR}}(X/D) \mid \dim H^i(X, E \otimes \Omega_X^\bullet(\log D)) \geq k\}.$$

Given any point  $\rho_0$  in  $\Sigma_k^i(U)$ , one can always find an extension  $(E_0, \nabla_0) \in \mathbf{M}_{\text{DR}}(X/D)$  of  $L_{\rho_0}$ , which is not in  $\mathbf{BL}$ , e.g., the Deligne extension. Then  $RH(\Sigma_k^i(X/D) - \mathbf{BL}) = \Sigma_k^i(U)$ .

Now, given  $T \subset \Sigma_k^i(U)$  as an irreducible component, take any point  $\rho_0$  in  $T$ . Since  $RH$  is analytically a covering map, there is a unique irreducible component  $S$  of  $RH^{-1}(T)$  containing the Deligne extension  $(E_0, \nabla_0)$  of  $L_{\rho_0}$ . Since  $S \not\subset \mathbf{BL}$  and since  $RH$  is analytically a covering map, we have  $RH(S) = T$ . By semicontinuity theorem,  $\Sigma_k^i(X/D) \subset \mathbf{M}_{\text{DR}}(X/D)$  is closed and defined over  $\bar{\mathbb{Q}}$ . Since  $S$  is an irreducible component of  $\Sigma_k^i(X/D)$ ,  $S$  is closed and defined over  $\bar{\mathbb{Q}}$ .  $\square$

Now, we are ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* Let  $T$  be an irreducible component of  $\Sigma_k^i(U)$ . By [DP, Lemma 9.2],  $\Sigma_k^i(U)$  is defined by some Fitting ideal coming from the CW-complex structure of  $U$ . Thus,  $\Sigma_k^i(U) \subset \mathbf{M}_B(U)$  is defined over  $\mathbb{Q}$ . Hence, as an irreducible component of  $\Sigma_k^i(U)$ ,  $T$  is defined over  $\bar{\mathbb{Q}}$ . According to Lemma 2.4, there exists  $S \subset \mathbf{M}_{\text{DR}}(X/D)$  defined over  $\bar{\mathbb{Q}}$  such that  $RH(S) = T$ . Then  $\text{res}(S) \subset \mathbb{C}^n$  and  $\text{ev}(T) \subset (\mathbb{C}^*)^n$  are defined over  $\bar{\mathbb{Q}}$ , and moreover,  $\exp(\text{res}(S)) = \text{ev}(T)$ . According to Lemma 2.2,  $\text{ev}(T)$  contains a torsion point  $\tau$ .

Since  $\tau \in (\mathbb{C}^*)^n$  is torsion, we can take  $l \in \mathbb{Z}_+$  such that  $\tau^l = 1 \in (\mathbb{C}^*)^n$ . Then the image of the  $l$ -power map  $(\cdot)^l : \text{ev}^{-1}(\tau) \rightarrow \mathbf{M}_B(U)$  is equal to  $\mathbf{M}_B(X)$ . Choose  $\eta \in \text{ev}^{-1}(\tau)$ , such that  $\eta^l = 1$ . Every  $\xi \in RH^{-1}(\eta)$  is a  $\bar{\mathbb{Q}}$  point in  $\mathbf{M}_{\text{DR}}(X/D)$ . In fact, since  $\eta^l = 1$  in  $\mathbf{M}_B(U)$ ,  $\xi^l$  is in the image of  $\mathbb{Z}^n \rightarrow \mathbf{M}_{\text{DR}}(X/D)$ . Recall that the image of  $\{m_\alpha\}_{1 \leq \alpha \leq n}$  under  $\mathbb{Z}^n \rightarrow \mathbf{M}_{\text{DR}}(X/D)$  is  $\bigotimes_{1 \leq \lambda \leq n} (\mathcal{O}_X(-D_\lambda), d)^{\otimes m_\lambda}$ , which is clearly a  $\bar{\mathbb{Q}}$  point in  $\mathbf{M}_{\text{DR}}(X/D)$ . Therefore,  $\xi$ , as an  $l$ -th root of a  $\bar{\mathbb{Q}}$  point, has to be a  $\bar{\mathbb{Q}}$  point.

Notice that

$$(\text{ev} \circ RH)^{-1}(\tau) = \bigcup_{\xi \in RH^{-1}(\eta)} (\xi \cdot \mathbf{M}_{\text{DR}}(X)).$$

Moreover, since  $RH(S) = T$ ,

$$\begin{aligned} T \cap \text{ev}^{-1}(\tau) &= RH(S) \cap \text{ev}^{-1}(\tau) \\ &= RH(S \cap (RH \circ \text{ev})^{-1}(\tau)) \\ &= \bigcup_{\xi \in RH^{-1}(\eta)} RH(S \cap (\xi \cdot \mathbf{M}_{\text{DR}}(X))). \end{aligned}$$

Each  $RH(S \cap (\xi \cdot \mathbf{M}_{\text{DR}}(X)))$  is closed in  $\mathbf{M}_B(U)$ , and  $T \cap \text{ev}^{-1}(\tau)$  is a noetherian topological space. Hence, for some  $\xi_0 \in RH^{-1}(\eta)$ ,  $RH(S \cap (\xi_0 \cdot \mathbf{M}_{\text{DR}}(X)))$  contains an

irreducible component of  $T \cap ev^{-1}(\tau)$ . Since  $RH(\xi_0) = \eta$ ,  $RH((\xi_0^{-1} \cdot S) \cap \mathbf{M}_{\text{DR}}(X))$  contains an irreducible component of  $\eta^{-1} \cdot (T \cap ev^{-1}(\tau))$ . Recall that  $\eta \in ev^{-1}(\tau)$ . Thus,  $\eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subset \mathbf{M}_B(X)$ .

Now,  $RH$  maps an irreducible component  $W$  of  $(\xi_0^{-1} \cdot S) \cap \mathbf{M}_{\text{DR}}(X)$  to an irreducible component  $RH(W)$  of  $\eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subset \mathbf{M}_B(X)$ . Both of these irreducible components are defined over  $\bar{\mathbb{Q}}$ . Indeed, since  $\xi_0$  and  $S$  are defined over  $\bar{\mathbb{Q}}$  in  $\mathbf{M}_{\text{DR}}(X/D)$ , and  $\eta$ ,  $T$ ,  $ev^{-1}(\tau)$  are defined over  $\bar{\mathbb{Q}}$  in  $\mathbf{M}_B(U)$ ,  $(\xi_0^{-1} \cdot S) \cap \mathbf{M}_{\text{DR}}(X)$  and  $\eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subset \mathbf{M}_B(X)$  are defined over  $\bar{\mathbb{Q}}$  in  $\mathbf{M}_{\text{DR}}(X/D)$  and  $\mathbf{M}_B(U)$ , respectively. Hence, the same is true for their irreducible components. Thus, we can apply [S2, Theorem 3.3] which says that this irreducible component  $RH(W)$  of  $\eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subset \mathbf{M}_B(X)$  is a torsion translate of a subtorus. In particular,  $\eta^{-1} \cdot (T \cap ev^{-1}(\tau)) \subset \mathbf{M}_B(X)$  contains a torsion point. Since  $\eta$  is also a torsion point,  $T$  must contain a torsion point.  $\square$

### 3. FINITE ABELIAN COVERS

First, we consider a more general situation. Let  $U$  be a connected, finite-type CW-complex, and let  $\mathbf{M}_B(U) = \text{Hom}(\pi_1(U), \mathbb{C}^*)$  be the moduli space of rank one local systems on  $U$ , which is naturally an algebraic group. Suppose  $\tau \in \mathbf{M}_B(U)$  is a torsion point. Denote the universal cover of  $U$  by  $\tilde{U}$ , and let  $H$  be the kernel of  $\tau : \pi_1(U) \rightarrow \mathbb{C}^*$ . Then  $H$  acts on  $\tilde{U}$  and we denote the quotient  $\tilde{U}/H$  by  $V$ . Now,  $\langle \tau \rangle$ , the subgroup of  $\mathbf{M}_B(U)$  generated by  $\tau$ , acts on  $V$ , and the quotient  $V/\langle \tau \rangle = U$ . Denote this quotient may by  $f : V \rightarrow U$ . Composing with  $f_* : \pi_1(V) \rightarrow \pi_1(U)$ ,  $f$  induces a homomorphism of algebraic groups  $f^* : \mathbf{M}_B(U) \rightarrow \mathbf{M}_B(V)$ . Under this construction, we immediately have  $f^*(\tau) = 1 \in \mathbf{M}_B(V)$  is the identity element, i.e.,  $f^*(\tau)$  maps every element in  $\pi_1(V)$  to 1. The main result of this section is the following:

**Proposition 3.1.** *Fixing  $i$ , suppose that for every  $k \in \mathbb{Z}_+$ , each irreducible component of  $\Sigma_k^i(V)$  containing 1 is a subtorus. Then for every  $k \in \mathbb{Z}_+$ , each irreducible component of  $\Sigma_k^i(U)$  containing  $\tau$  is a translate of subtorus.*

*Proof.* Denote the order of  $\tau$  in  $\mathbf{M}_B(X)$  by  $r$ . For any local system  $L$  on  $U$ ,

$$f_* f^*(L) \cong \bigoplus_{j=0}^{r-1} L \otimes_{\mathbb{C}} L_{\tau}^{\otimes j}.$$

According to the projection formula,  $H^i(V, f^*(L)) \cong H^i(U, f_* f^*(L))$ . Therefore,

$$(1) \quad \dim H^i(V, f^*(L)) = \sum_{j=0}^{r-1} \dim H^i(U, L \otimes L_{\tau}^{\otimes j}).$$

Let  $T$  be an irreducible component of  $\Sigma_k^i(U)$  containing  $\tau$ , and let  $\rho$  be a general point in  $T$ . Define  $\beta_j = \dim H^i(U, L_{\rho} \otimes L_{\tau}^{\otimes j})$ , for  $0 \leq j \leq r-1$ , and  $\beta = \sum_{0 \leq j \leq r-1} \beta_j$ . It is possible that  $T \subset \Sigma_{k+1}^i(U)$ , and in this case,  $\beta > k$ .

**Claim.**  $f^*(T)$  is an irreducible component of  $\Sigma_{\beta}^i(V)$ .

*Proof of Claim.* By the definition of  $\rho$  and  $\beta$ , it is clear that  $f^*(T) \subset \Sigma_\beta^i(V)$ . Let  $S$  be the irreducible component of  $\Sigma_\beta^i(V)$  containing  $f^*(T)$ . We want to show that  $S = f^*(T)$ . Let  $\tilde{S}$  be a connected component of  $(f^*)^{-1}(S)$  containing  $T$ . Since  $f^*$  is a covering map,  $\tilde{S}$  is irreducible and is a covering space of  $S$ .

Suppose  $T \subsetneq \tilde{S}$ . Take a general point  $\rho'$  in  $\tilde{S}$ . Since  $\tilde{S}$  is irreducible, and since  $T$  is an irreducible component of  $\Sigma_k^i(U)$ , we can assume  $\rho' \notin \Sigma_k^i(U)$ . Therefore,  $\dim H^i(U, L_{\rho'}) < \dim H^i(U, L_\rho)$ . Since  $\rho'$  is more general than  $\rho$ ,  $\dim H^i(U, L_{\rho'} \otimes L_\tau^{\otimes j}) \leq \dim H^i(U, L_\rho \otimes L_\tau^{\otimes j})$ , for every  $1 \leq j \leq r-1$ . Thus,

$$\sum_{j=0}^{r-1} \dim H^i(U, L_{\rho'} \otimes L_\tau^{\otimes j}) < \sum_{j=0}^{r-1} \dim H^i(U, L_\rho \otimes L_\tau^{\otimes j}).$$

Now, equality (1) implies that  $\dim H^i(V, f^*(L_{\rho'})) < \beta$ , and hence  $f^*(\rho')$  is not contained in  $\Sigma_\beta^i(V)$ . This is a contradiction to the definition of  $\rho'$  and  $\tilde{S}$ . So we have proved  $T = \tilde{S}$ . Therefore,  $f^*(T) = S$  is an irreducible component of  $\Sigma_\beta^i(V)$ .  $\square$

Since  $\tau \in T$ ,  $f^*(T)$  contains 1. By the assumption of the theorem,  $f^*(T)$  is a subtorus in  $\mathbf{M}_B(V)$ . Since  $f^*$  is a covering map, obviously  $T$  must be a translate of subtorus. We finished the proof of the proposition.  $\square$

Theorem 1.1 is a direct consequence of Theorem 2.1, Proposition 3.1, and Theorem 1.2.

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